Convolución.

Sean f, g SCOE tales que $f(t) = g(t) = 0$ si $t < 0$

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-y)\, g(y)\, dy$$ si $y<0$, $g(y)=0$

$$= \int_{0}^{\infty} f(t-y)\, g(y)\, dy$$ si $t-y<0$, $f(t-y)=0$; $y>t$

$$= \int_{0}^{t} f(t-y)\, g(y)\, dy$$

$$\Rightarrow \boxed{(f*g)(t) = \begin{cases} \int_0^t f(t-y)\, g(y)\, dy & \text{si } t \geq 0 \\ 0 & \text{si } t<0 \end{cases}}$$

Propiedades:
- $f*g = g*f$
- $(f+g)*h = f*h + g*h$
- $(\lambda f)*g = \lambda (f*g)$
- $(f*g)*h = f*(g*h)$
- f*g es de orden exponencial, y continua

Ejemplo. $f(t) = e^t$
$g(t) = t$

$$f*g(t) = \int_0^t f(t-y)\, g(y)\, dy = \int_0^t e^{t-y}\, y\, dy = e^t \int_0^t e^{-y} \cdot y\, dy =$$

$$= e^t \cdot \left[-y e^{-y} \Big|_0^t + \int_0^t 1 \cdot e^{-y}\, dy \right] = e^t \left(-t e^{-t} - e^{-t} + 1 \right) =$$

$$= -t - 1 + e^t, \quad \underline{t \geq 0}$$

Teorema

Sean f, g SCOE, entonces.

i) $f * g$ es de Orden exponencial

ii) $\mathcal{L}(f*g)(s) = \mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s)$

Dem:
$$|f(t)| \leq M_1 e^{\alpha t}$$
$$|g(t)| \leq M_2 e^{\beta t}$$

$$|(f*g)(t)| = \left|\int_0^t f(t-y)\, g(y)\, dy\right| \leq \int_0^t |f(t-y)|\, |g(y)|\, dy$$
$$\leq \int_0^t M_1 e^{\alpha(t-y)} \cdot M_2 e^{\beta y}\, dy = M_1 M_2 e^{\alpha t} \int_0^t e^{y(\beta-\alpha)}\, dy.$$

Si $\beta \neq \alpha$:, sea $\beta > \alpha$
$$|f*g|(t) \leq M_1 M_2 e^{\alpha t}\left(\frac{e^{t(\beta-\alpha)}}{\beta-\alpha} - \frac{1}{\beta-\alpha}\right) = \frac{M_1 M_2}{\beta-\alpha}\left(e^{t\beta} - e^{t\alpha}\right) \leq \frac{M_1 M_2}{\beta-\alpha} e^{t\beta}$$

Si $\beta = \alpha$:
$$|f*g|(t) \leq M_1 M_2 e^{\alpha t} \cdot t \leq M_1 M_2 M_3 e^{(\alpha+\varepsilon)t}$$
$$\downarrow$$
$t \leq M_3 e^{\varepsilon t}$, para c/ $\varepsilon > 0$.

$\Rightarrow f*g$ es de orden exponencial $\Rightarrow$ tiene T.L.

$$\mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s) = \int_0^\infty e^{-st} f(t)\, dt \cdot \int_0^\infty e^{-sy} g(y)\, dy =$$
$$= \int_0^\infty \int_0^\infty e^{-st} e^{-sy} f(t) \cdot g(y)\, dy\, dt =$$

$u = t + y$:
$du = dy$
$$= \int_0^\infty \int_t^\infty e^{-su} f(t)\, g(u-t)\, du\, dt$$

Si hacemos $g(\tau)=0$ para $\tau<0 \Rightarrow g(u-t)=0$ si $u<t$

$$\mathcal{L}(f)(s)\cdot\mathcal{L}(g)(s) = \int_0^\infty \int_0^\infty e^{-su} f(t)\, g(u-t)\, du\, dt$$

Invirtiendo orden de integración:

$f(t)=0$ si $t<0$

$g(u-t)=0$ si $t>u$

$$= \int_0^\infty \left(\int_0^\infty f(t)\, g(u-t)\, dt \right) e^{-su} du$$

$$= \int_0^\infty \left(\int_0^u f(t)\, g(u-t)\, dt \right) e^{-su} du$$

$$= \int_0^\infty (f*g)(u)\, e^{-su} du$$

$$= \mathcal{L}\left(f*g\right)(s)$$

Ejemplo:

Hallar la antitransformada de $F(s) = \frac{1}{(1+s^2)^2}$

Vemos que $F(s) = G(s)\cdot G(s)$ con $G(s) = \frac{1}{1+s^2} = \mathcal{L}(g(t))(s)$, $g(t) = \operatorname{sen} t$

$$\Rightarrow f(t) = \mathcal{L}^{-1}(F(s))(t) = (g*g)(t) = \int_0^t g(y)\, g(t-y)\, dy = \int_0^t \operatorname{sen}(y)\operatorname{sen}(t-y)\, dy$$

$$= \int_0^t \operatorname{sen}(y)(\operatorname{sen} t \cos y - \cos t \operatorname{sen} y)\, dy =$$

$$= \int_0^t \operatorname{sen} t \cdot \cos y \operatorname{sen} y\, dy - \int_0^t \cos t \operatorname{sen}^2 y\, dy =$$

$$= \operatorname{sen} t \cdot \frac{\operatorname{sen}^2 y}{2}\Big|_0^t - \cos t \cdot \frac{1}{2}(y - \operatorname{sen}(y)\cos(y))\Big|_0^t =$$

$$= \frac{\operatorname{sen}^3 t}{2} - \frac{t}{2}\cos t + \frac{\cos^2 t \operatorname{sen} t}{2} = \frac{1}{2}\operatorname{sen} t\,(\operatorname{sen}^2 t + \cos^2 t) - \frac{t}{2}\cos t$$

$$= \frac{1}{2}\operatorname{sen} t - \frac{t}{2}\cos t$$

Antitransformar F(s).

Ejemplo: $F(s) = \dfrac{1}{s^3(s-1)} = H(s) \cdot G(s)$

con $H(s) = \dfrac{1}{s^3} = \mathcal{L}(h(t))(s)$ siendo $h(t) = \dfrac{t^2}{2}$, $t>0$

$G(s) = \dfrac{1}{s-1} = \mathcal{L}(g(t))(s)$ siendo $g(t) = e^t$ $t>0$

$$\Rightarrow f(t) = \mathcal{L}^{-1}(F(s))(t) = h * g\,(t) = \int_0^t h(y)\,g(t-y)\,dy =$$

$$= \int_0^t \frac{y^2}{2} \cdot e^{t-y}\,dy = \frac{e^t}{2}\int_0^t y^2 e^{-y}\,dy$$

$$= \frac{e^t}{2} \cdot \left[-e^{-y}(y^2+2y+2)\right]_0^t = \frac{e^t}{2}\left[-e^{-t}(t^2+2t+2) + 2\right] =$$

$$\boxed{= -\frac{1}{2}(t^2+2t+2) + e^t}$$

Antitransformada de funciones racionales.

Recordamos: $\mathcal{L}(e^{at})(s) = \dfrac{1}{s-a}$

$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(t^n e^{at})(s) = \frac{n!}{(s-a)^{n+1}}$$

Sea $F(s) = \dfrac{P(s)}{Q(s)}$ con P y Q polinomios con $gr(P) < gr(Q)$

Supongamos que Q tiene k raíces distintas $s_1, s_2, \dots s_k$, de multiplicidad 1, que no son raíces de $P(s)$

¿Y si $F(s) = \frac{P(s)}{Q(s)}$ y los ceros de Q no son simples? Similar...

$\frac{as+b}{(s-1)^2}$

Ejemplo.

$$F(s) = \frac{2s^2}{(s-1)^2 \cdot (s+2)} = \frac{A_1}{s-1} + \frac{A_2}{(s-1)^2} + \frac{A_3}{s+2}$$

-2 es polo simple:

$$\lim_{s\to -2} (s+2) F(s) = \lim_{s\to -2} \frac{2s^2}{(s-1)^2} = \boxed{\frac{8}{9} = A_3}$$

1 es polo doble.

$$\lim_{s\to 1} (s-1)^2 F(s) = \lim_{s\to 1} (s-1)A_1 + A_2 + (s-1)^2 \frac{A_3}{(s+2)} = A_2$$

$$\lim_{s\to 1} \frac{2s^2}{s+2} = \boxed{\frac{2}{3} = A_2}$$

Finalmente, $A_1 = \operatorname{Res}(F, 1) = \lim_{s\to 1} \left((s-1)^2 F(s)\right)' =$

$$= \lim_{s\to 1} \left(\frac{2s^2}{s+2}\right)' = \lim_{s\to 1} \frac{4s(s+2) - 2s^2}{(s+1)^2} = \boxed{\frac{10}{9} = A_1}$$

Luego: $F(s) = \frac{10}{9} \frac{1}{s-1} + \frac{2}{3} \frac{1}{(s-1)^2} + \frac{8}{9} \cdot \frac{1}{s+2}$

$$\boxed{f(t) = \mathcal{L}^{-1}(F)(t) = \frac{10}{9} e^{t} + \frac{2}{3} t e^{t} + \frac{8}{9} e^{-2t}}, \quad t>0$$

4 bis

Entonces (fracciones simples)

$$F(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_k}{s-s_k}$$

Quiénes son A_j? Noten, s_j son polos simples

$$\lim_{s\to s_j} (s-s_j)F(s) = A_j = \text{Res}(F, s_j)$$

$$\Rightarrow F(s) = \sum_{j=1}^{k} \frac{\text{Res}(F,s_j)}{s-s_j}$$

$\mathcal{L}^{-1}$ (

$$f(t) = \mathcal{L}^{-1}(f)(t) = \sum_{j=1}^{k} \text{Res}(F,s_j)\cdot e^{s_j t} \quad t>0$$

Además: $A_j = \lim_{s\to s_j} (s-s_j)\cdot\frac{P(s)}{Q(s)} \overset{L'H}{=} \lim \frac{P(s)+(s-s_j)P'(s)}{Q'(s)} = \frac{P(s_j)}{Q'(s_j)}$

$$\boxed{f(t) = \mathcal{L}^{-1}(f)(t) = \sum_{j=1}^{k} \frac{P(s_j)}{Q'(s_j)}\cdot e^{s_j t}}, \; t>0$$

Ejemplo Antitransformar $F(s) = \frac{1}{s^3+3s^2+2s}$

$F(s) = \frac{1}{s(s+2)(s+1)} \rightarrow P(s)=1$

$Q(s) = s^3+3s^2+2s \qquad Q'(s) = 3s^2+6s+2$

$s_1 = 0$

$s_2 = -2$

$s_3 = -1$

$$\rightarrow \boxed{f(t) = \mathcal{L}^{-1}(f)(t) = \frac{1}{2}e^{0t} + \frac{1}{2}e^{-2t} + e^{-t}}, \; t>0$$

$$\frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$$

Transformada de funciones periódicas, periodo L.

Sea $f(t) = \begin{cases} f_1(t) & 0 \le t \le L \\ f_1(t-nL) & nL \le t \le (n+1)L \end{cases}$

$SCOE_0$

$$\mathcal{L}(f)(s) = \int_0^\infty f(t) e^{-st} dt =$$

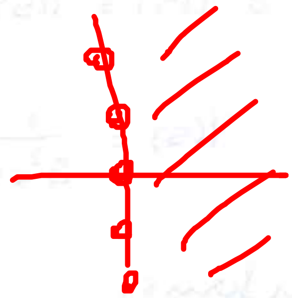
$$= \int_0^L f(t) e^{-st} dt + \int_L^\infty \underbrace{f(t) e^{-st} dt}_{u = t-L} = \int_0^L \cancel{f_1(t) e^{-st} dt + \int_0^\infty} \underbrace{f(u+L)}_{= f(u)} e^{-s(u+}$$

$$= \int_0^L f_1(t) e^{-st} dt + \int_0^\infty f(u) e^{-s(u+L)} du = \int_0^L f_1(t) e^{-st} dt + e^{-sL} \int_0^\infty f(u) e^{-su} du$$

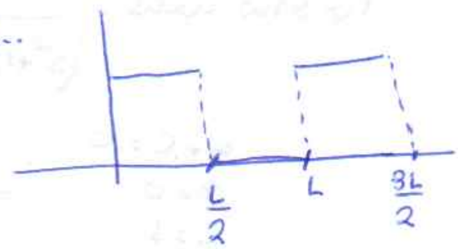
$$F(s) = \int_0^L f_1(t) e^{-st} dt + e^{-sL} F(s)$$

$$F(s) \cdot (1 - e^{-sL}) = \int_0^L f_1(t) e^{-st} dt$$

$$F(s) = \frac{\int_0^L f_1(t) e^{-st} dt}{1 - e^{-sL}} \longrightarrow = 0 \quad sL = 2\pi k i$$



Ejemplo: $f(t) = \begin{cases} 1 & nL \le t \le nL + \frac{L}{2} \quad n = 0, 1, 2, \ldots \\ 0 & nL + \frac{L}{2} \le t \le (n+1)L \end{cases}$



Periodo L.

$$\int_0^L f_1(t) e^{-st} dt = \int_0^{L/2} e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_0^{L/2} = \frac{e^{-s\frac{L}{2}} - 1}{-s} = \frac{1 - e^{-s\frac{L}{2}}}{s}$$

$$F(s) = \frac{1 - e^{-s\frac{L}{2}}}{\underbrace{s(1 - e^{-sL})}_{1^2 - \left(e^{-\frac{sL}{2}}\right)^2}} = \frac{1 - e^{-s\frac{L}{2}}}{s(1 - e^{-s\frac{L}{2}})(1 + e^{-s\frac{L}{2}})} = \frac{1}{s(1 + e^{-sL/2})}$$

Resolución de PVI con T.L.

Recordando: $\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0+)$

$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s\,f(0+) - f'(0+)$

...

Ejemplo. Hallar $y(t)$ tal que:

$$\mathcal{L}\left(\begin{cases} y'' + y = 8\,\underbrace{H_a(t)}_{H(t-a)} = \begin{cases} 8 & t>a \\ 0 & t<a \end{cases} & (a>0) \\ y(0)=0 \\ y'(0)=1 \end{cases}\right.$$

$$s^2 Y(s) - s\overset{0}{y(0)} - \overset{1}{y'(0)} + Y(s) = \mathcal{L}(8H_a(t))(s) = 8\cdot\frac{e^{-as}}{s}$$

$$s^2 Y(s) + Y(s) = \frac{8e^{-as}}{s} + 1$$

$$Y(s) = \frac{1}{s^2+1}\cdot\left(\frac{8e^{-as}}{s} + 1\right) = 8\cdot\frac{1\,e^{-as}}{(s^2+1)s} + \frac{1}{s^2+1}$$

Antitransformar:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \operatorname{sen} t$$

Por otro lado: $$\frac{1}{(s^2+1)s} = \frac{A}{s-i} + \frac{B}{s+i} + \frac{C}{s} \overset{o\ldots}{\underset{\uparrow}{=}} \frac{\alpha s+\beta}{s^2+1} + \frac{C}{s} = \frac{\alpha s^2+\beta s+Cs^2+C}{(s^2+1)s}$$

$$\begin{cases}\alpha + C = 0 \\ \beta = 0 \\ C = 1\end{cases} \Rightarrow \alpha = -1,\ \beta = 0,\ C = 1$$

$$A = \operatorname{Res}\left(\frac{1}{(s^2+1)s}, i\right)$$

$$\frac{1}{(s^2+1)s} = -\frac{s}{s^2+1} + \frac{1}{s}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)s}\right)(t) = -\cos(t) + 1,\quad t>0$$

$$\underbrace{Ae^{it} + Be^{-it}}_{\cos t} + C$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{e^{-as}}{(s^2+1)s}\right)(t) = \left(-\cos(t-a)+1\right)H(t-a)$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{8e^{-as}}{(s^2+1)s} + \frac{1}{s^2+1}\right) = \left(8-8\cos(t-a)\right)H(t-a) + \operatorname{sen}(t)\,H(t)$$

Ejemplo:

$$\begin{cases} y''+y = \begin{cases} \operatorname{sen} t & 0<t<\pi \\ 0 & t>\pi \end{cases} \\ y(0)=0 \\ y'(0)=0 \end{cases}$$

$\mathcal{L}$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \int_0^{\pi} \operatorname{sen} t\, e^{-st}\,dt$$

$$(s^2+1)\,Y(s) = \int_0^{\pi} \frac{e^{it}-e^{-it}}{2i}\, e^{-st}\,dt =$$

$$= \frac{1}{2i}\left(\frac{e^{(i-s)t}}{i-s} - \frac{e^{(-i-s)t}}{-i-s}\right)\Bigg|_0^{\pi} =$$

$$= \frac{1}{2i}\left(\frac{e^{(i-s)\pi}-1}{i-s} - \frac{(e^{(-i-s)\pi}-1)}{-i-s}\right)$$

$$= \frac{1}{2i}\left(-e^{-s\pi}\left(\frac{1}{i-s}+\frac{1}{i+s}\right) + \frac{1}{s-i} - \frac{1}{s+i}\right)$$

$$= \frac{1}{2i}\left(\frac{e^{-s\pi}\cdot 2i}{s^2+1} + \frac{2i}{s^2+1}\right)$$

$$(s^2+1)\,Y(s) = \frac{1}{s^2+1}\left(e^{-s\pi}+1\right)$$

$$Y(s) = \frac{1}{(s^2+1)^2}\left(e^{-s\pi}+1\right) = \frac{e^{-s\pi}}{(s^2+1)^2} + \frac{1}{(s^2+1)^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)(t) = \text{sen}\,t * \text{sen}\,t \underset{\downarrow}{=} \frac{1}{2}(\text{sen}\,t - t\cos t),\ t>0$$

pág 3

$$\Rightarrow y(t) = \frac{1}{2}\left(\text{sen}(t-\pi) - (t-\pi)\cos(t-\pi)\right)H(t-\pi) + \frac{1}{2}(\text{sen}\,t - t\cos t)H(t)$$

$$= \frac{1}{2}\left(-\text{sen}\,t + (t-\pi)\cos t\right)H(t-\pi) + \frac{1}{2}(\text{sen}\,t - t\cos t)H(t)$$

$$y(t) = \begin{cases} 0 & t<0 \\ \frac{1}{2}\text{sen}\,t - t\cos t & 0<t<\pi \\ \frac{1}{2}(\text{sen}\,t - t\cos t) - \frac{1}{2}\text{sen}\,t + \frac{1}{2}(t-\pi)\cos t & t>\pi \end{cases}$$

$$y(t) = \begin{cases} 0 & t<0 \\ \frac{1}{2}(\text{sen}\,t - t\cos t) & 0<t<\pi \\ -\frac{\pi}{2}\cos t & t>\pi \end{cases}$$

Ejemplo Hallar $x(t)$ e $y(t)$ tales que:

$$\begin{cases} y' + x' + y + x = 1 \\ y' + x = e^t \\ y(0) = -1 \\ x(0) = 2 \end{cases}$$

$$\mathcal{L}\Big(\quad \begin{cases} sY(s) - y(0) + sX(s) - x(0) + Y(s) + X(s) = \frac{1}{s} \\ sY(s) - y(0) + X(s) = \frac{1}{s-1} \end{cases}$$

$$\begin{cases} X(s)\cdot(s+1) + Y(s)(s+1) = \frac{1}{s} - 1 + 2 = \frac{1}{s} + 1 = \frac{s+1}{s} \\ X(s) + Y(s)\cdot s = \frac{1}{s-1} - 1 = \frac{1-s+1}{s-1} = \frac{2-s}{s-1} \end{cases}$$

$$\begin{bmatrix} s+1 & s+1 \\ 1 & s \end{bmatrix}\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} \frac{s+1}{s} \\ \frac{2-s}{s-1} \end{pmatrix}$$

θ : $\begin{bmatrix} 1 & 1 \\ 1 & s \end{bmatrix} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s} \\ \frac{2-s}{s-1} \end{pmatrix}$

$$X(s) + Y(s) = \frac{1}{s}$$
$$X(s) + sY(s) = \frac{2-s}{s-1}$$

$\Rightarrow$ $Y(s)(s-1) = \frac{2-s}{s-1} - \frac{1}{s} = \frac{2s - s^2 - s + 1}{(s-1)s} = \frac{s+1-s^2}{(s-1)s}$

Res$(Y(s), 0)$ Res$(Y, 1)$

$$Y(s) = \frac{s+1-s^2}{(s-1)^2 s} = \frac{1}{s} - \frac{2}{s-1} + \frac{1}{(s-1)^2}$$

$\downarrow$ fracc. simples $\lim_{s \to 1} (s-1)^2 Y(s)$

$\Rightarrow$ $X(s) = \frac{1}{s} - Y(s) = \frac{2}{s-1} - \frac{1}{(s-1)^2}$

$\Rightarrow \mathcal{L}^{-1}$: $x(t) = (2e^t - te^t) H(t)$

$$y(t) = (1 - 2e^t + te^t) H(t)$$